Independence of Families of Solutions of Ordinary Differential Equations with Constant Coefficients

We have already discussed how there are naturally arising families of functions of certain types which are solutions of equations of the form

$$(D-\lambda_1)^{k_1}\dots(D-\lambda_\ell)^{k_\ell}\, y = 0 \qquad (*)$$

where $k_i \geq 1$ and the λ_i's are distinct.

Namely, if λ_i is real, then $(D-\cancel{\alpha}\lambda_i)^{k_i}\,(x^p e^{\lambda_i x}) = 0$ if $0 \leq p \leq k_i - 1$. [This is easily checked by using that

$$(D-\lambda_i)(x^q e^{\lambda_i x}) = q\, x^{q-1} e^{\lambda_i x} + \lambda_i\, x^q e^{\lambda_i x} - \lambda_i x^q e^{\lambda_i x}$$
$$= q\, x^{q-1} e^{\lambda_i x}$$

so that $(D-\lambda_i)^{k_i}(x^p e^{\lambda_i x}) = 0$ if $p = 0, \dots, k_i - 1$.]

If λ_i is complex, the same applies with the usual definition that $e^{a+bi} = e^a(\cos b + i \sin b)$: same proof.

Thus one obtains $k_1 + \dots + k_\ell$ solutions of $(*)$ namely (for real λ's) $\quad x^p e^{\lambda_i x} \quad 0 \leq p \leq k_i - 1$

and for complex λs, $\lambda_i = a_i + i b_j$ and associated $\bar{\lambda}_i$

$x^p e^{a_i x} \cos b_i x$ $\qquad$ which the k for $\bar{\lambda}_i$ the

$x^p e^{a_i x} \sin b_i x$ $\qquad$ same as for λ_i

Thus there are $k_1 + \dots + k_\ell$ solutions (where $k_1 + \dots + k_\ell$ = order of the equations). If these are "independent" then the general solution of the equation $(*)$ is a (real) constant coefficient linear combination of them since it follows

for the Existence and Uniqueness Theorem that the dimension of the vector space of solutions of (*) is exactly $k_1 + \cdots + k_\ell$. [Recall the logic:
Set $n = k_1 + \cdots + k_\ell$
Sending y to $\left(y(0), \frac{dy}{dx}\Big|_0, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\Big|_0\right)$

is an isomorphic (1-1 onto) linear transformation of the space of solutions onto $\mathbb{R}^n$, by E&U Th.].

Our goal is to prove that the specific solutions listed are in fact linearly independent, i.e., a linear combination of them with real constant coefficients that is $\equiv 0$ must have all the coefficients $= 0$.

To prove this, suppose such a linear combination that is $\equiv 0$ is given. Let a_{max} = the largest value of λ_i real or real part of λ_i if λ_i is complex that occurs with a nonzero coefficient, say $C \neq 0$
Then
with $C \neq 0$

$C e^{a_{max} x}$ (linear combination of 1, x powers and (x powers)(cos bx), and (x powers)(sin bx), various b values)

= linear combination of terms like
e^{ax} (x powers) or e^{ax}(x powers) cos bx
or e^{ax}(x powers) sin bx, various b's,
where $a < a_{max}$.

Multiplying by $e^{-a_{max} x}$ gives

the thing on the LHS above

$$\lim_{x \to \infty} \Big(\text{linear combination of } 1, \ x \text{ powers}, \ (x \text{ powers}) \cos bx, \ (x \text{ powers}) \sin bx \Big)$$

$= 0$

since $e^{ax - a_{max} x}$ (any x power) $\to 0$ as $x \to +\infty$

If we can show that $\lim_{x \to +\infty}$ (linear comb. here) $= 0$

$\Rightarrow$ linear combination has zero coefficients,

then we are done, by repeated application of the argument (the $e^{a_{max} x}$ part $= 0$, so then we do the next smallest e^{ax} occurring, etc.).

Thus we need to show that: $p_0(x) + p_1(x) \cos b_1 x$

$+ q_1(x) \sin b_1 x + p_2(x) \cos b_2 x + q_2(x) \sin b_2 x +$

$+ p_\ell(x) \cos b_\ell(x) + q_\ell(x) \sin b_\ell(x)$ has limit 0 as $x \to +\infty$

$\Rightarrow$ all p's, q's are zero polynomials.

(given that the p's & q's are polynomials and the $b_1, \ldots, b_\ell$ are all different from each other).

[Note: ℓ here is not the same ℓ as earlier: It can be any nonnegative integer].

We prove this using the concept of "average value":

Definition: The average value of a continuous function f on an interval $[\alpha,\beta]$ =

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\,dx. \quad \text{Notation } A_f([\alpha,\beta])$$

Note that:

(1) $\lim_{x\to+\infty} f(x) = 0 \Rightarrow \lim_{\alpha\to+\infty} A_f([\alpha,\beta]) = 0$

where β can be taken arbitrarily as long as $\beta>\alpha$.
Namely, for each $\varepsilon>0$, $\exists\ \alpha(\varepsilon)$ such that

$$|A_f([\alpha,\beta])| < \varepsilon \quad \text{if} \quad \beta>\alpha>\alpha(\varepsilon).$$

Proof: Choose $\alpha(\varepsilon)$ such that $|f(x)|<\varepsilon$ if $x>\alpha(\varepsilon)$. Then for $\beta>\alpha>\alpha(\varepsilon)$

$$|Av_f([\alpha,\beta])| \le \left|\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\,dx\right| \le \frac{1}{\beta-\alpha}\cdot(\beta-\alpha)\max_{[\alpha,\beta]}|f|$$

$$< \frac{1}{\beta-\alpha}\cdot \beta-\alpha\cdot\varepsilon = \varepsilon \quad \text{since } |f(x)|<\varepsilon \text{ on } [\alpha,+\infty).$$

$\square$

(2) $A_{f_1+f_2}([\alpha,\beta]) = A_{f_1}([\alpha,\beta]) + A_{f_2}([\alpha,\beta])$

(3) $\lim_{\alpha\to+\infty} A_{\sin Bx}([\alpha,2\alpha]) = 0$ any fixed α, B

$\lim_{\alpha\to+\infty} A_{\cos Bx}([\alpha,2\alpha]) = 0$ any fixed α, B

(3) $\lim_{\alpha \to +\infty} A_{\sin^2 Bx}([\alpha, 2\alpha]) = \frac{1}{2}$

(4) $\lim_{\alpha \to +\infty} A_{\cos^2 Bx}([\alpha, 2\alpha]) = \frac{1}{2}$

Items (3) & (4) comes from doing the integrals.

Now consider the sum (p's polynomials) as before:

$p_0(x) + p_1(x) \cos b_1 x + q_1(x) \sin b_1 x + \cdots$

and suppose

$\lim = 0$ as $x \to +\infty$.

Collect all terms with maximum x exponent and lower exponents

$x^{\max} \left(C_0 + C_1 \cos b_1 x + D_1 \sin b_1 x + \cdots \right)$

some may be missing

$+ x^{\max - 1} \left(\quad \right) + x^{\max - 2} \left(\quad \right) + \cdots$

Multiplying by $x^{-\max}$ we get (from limit of whole thing $= 0$) that

$$\lim_{x \to +\infty} \left(C_0 + C_1 \cos b_1 x + D_1 \sin b_1 x + \cdots \right) = 0.$$

So now it remains only to show that

$$\lim_{x \to +\infty} \big(C_0 + C_1 \cos b_1 x + D_1 \sin b_1 x + \cdots \big) = 0$$

finite sum

$\Rightarrow \quad C_0 = 0, \; C_1 = 0, \; D_1 = 0$ etc.

To get $C_0 = 0$, note that

$$0 = \lim_{\alpha \to +\infty} A_{(C_0 + \dots)}[\alpha, 2\alpha]$$

$$= C_0 \quad \text{(since sine \& cosine terms have average value} \to 0\text{)}$$

So $C_0 = 0$.

To show, e.g. $C_1 = 0$ multiply by $\cos b_1 x$ (which has $|\cos b_1 x| \le 1$). Then

call this H

$$(**) \lim_{x \to +\infty} \left(0 \cos b_1 x + C_1 \cos^2 b_1 x + D_1 \sin b_1 x \cos b_1 x + C_2 \cos b_1 x \cos b_2 x + D_2 \cos b_1 x \sin b_2 x + \dots \right).$$

$= 0$

because $|\cos b_1 x| \le 1$ and hence

$\lim A_H([\alpha, 2\alpha]) = 0$ as $\alpha \to +\infty$.

Now we apply the following facts (proof later):

$$A_{\sin(b_1 x)\cos(b_1 x)}([\alpha, 2\alpha]) \to 0 \text{ as } \alpha \to +\infty$$

$$A_{\sin(b_2 x)\cos(b_1 x)}([\alpha, 2\alpha]) \to 0 \text{ as } \alpha \to +\infty$$

$$A_{\cos b_2 x \cos b_1 x}([\alpha, 2\alpha]) \to 0 \text{ as } \alpha \to +\infty.$$

Since $A_{\cos^2(b_1 x)}([\alpha, 2\alpha]) \to \frac{1}{2}$ as $\alpha \to +\infty$

we get from $(**)$ (all terms but the one with C_1 have $A(\text{term}) \to 0$)

$$\frac{1}{2} C_1 = 0 \quad \text{or} \quad C_1 = 0.$$

Similar reasoning gives that all other coefficients $= 0$.

It remains to check the average value statements: For this, note that

$$-2\sin b_1 x \sin b_2 x = \cos[(b_1+b_2)x] - \cos[(b_1-b_2)x]$$

$$2\sin b_1 x \cos b_2 x = \sin[(b_1-b_2)x] + \sin[(b_1+b_2)x]$$

$$2\cos b_1 x \cos b_2 x = \cos[(b_1+b_2)x] + \cos[(b_1-b_2)x].$$

Since everything on the right has average value on $[a, 2a] \to 0$ except $\cos(b_1-b_2)x$ when $b_1 = b_2$, the result follows. [Note: Here we are assuming without loss of generality that all b's > 0]